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# Characterization of $SU(1, 1)$ coherent states in terms of affine group wavelets

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## Abstract

The Perelomov coherent states of  $SU(1, 1)$  are labelled by elements of the quotient of  $SU(1, 1)$  by the compact subgroup. Taking advantage of the fact that this quotient is isomorphic to the affine group of the real line, we are able to parametrize the coherent states by elements of that group or equivalently by points in the half-plane. Such a formulation permits to find new properties of the  $SU(1, 1)$  coherent states and to relate them to affine wavelets.

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## 1. Introduction

Coherent states associated with the affine and  $SU(1, 1)$  groups have been introduced in different situations and have led to applications in fields of physics that are not directly connected. It is the purpose of the present work to exhibit the relations existing between some of those states.

The group of affine transformations of the real line plays an essential role in the analysis of acoustic and electromagnetic signals depending on one variable (e.g. the time). Systems of coherent states associated with that group, more recently known as wavelets, have been introduced as overcomplete bases of a Hilbert space in which a unitary irreducible representation acts [1, 2]. Their construction requires the choice of an admissible fiducial state (or mother wavelet) which is subsequently displaced by the operators of the representation under consideration. For a special choice of this basic state, the corresponding system of coherent states has minimal properties that have proved useful in applications [3].

The role of  $SU(1, 1)$  in physics, especially in quantum physics, has been recognized for a long time and its coherent states have been extensively studied. Due to the more complex structure of the group, several definitions are available even with the sole requirement of obtaining overcomplete bases [4–6]. Restricting to the systems of coherent states generated by displacement of a fundamental state, one still obtains different solutions, depending on the group representation and the initial state. To make a connection with the affine group coherent

states, we will consider only the discrete series representations acting on a rotation invariant basic state. As shown in [4], this choice leads to a system of coherent states labelled by the elements of the quotient of  $SU(1, 1)$  by the rotation group. The study could be adapted to the fundamental series of representations but the fiducial state must always have a rotation invariance.

The question of the comparison between the coherent states corresponding to the affine and  $SU(1, 1)$  groups arises because both appear in the problem of the Morse potential [7] and, more fundamentally, because the affine group is isomorphic to a subgroup of  $SU(1, 1)$ . In the following, we will establish the precise relation existing between the two sets of states and discuss the applications.

The study is most easily performed by realizing the discrete series representations of  $SU(1, 1)$  in spaces  $L_k^2(\mathbb{R}^+)$  of functions on the half-line in which irreducible representations of the affine group are naturally realized, as is recalled in section 2. In sections 3 and 4, we give explicit expressions for the canonical bases and the Perelomov coherent states in these spaces. In section 5, the latter states are parametrized in terms of the affine group. The identification of specific affine wavelets and the comparison with Morse states follow. New properties of the  $SU(1, 1)$  coherent states are obtained in section 6.

## 2. Unitary representations of the affine and $SU(1, 1)$ groups in spaces $L_k^2(\mathbb{R}^+)$

In this section, we recall useful formulae concerning the isomorphic groups  $SU(1, 1)$  and  $SL(2, \mathbb{R})$  and their affine subgroups. The group  $SU(1, 1)$  consists of matrices of the form

$$\Gamma = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \bar{\gamma}_2 & \bar{\gamma}_1 \end{pmatrix} \quad (1)$$

where  $\gamma_1, \gamma_2$  are complex numbers such that

$$|\gamma_1|^2 - |\gamma_2|^2 = 1. \quad (2)$$

The generators of its Lie algebra are

$$J_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad J_1 = \frac{-i}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad J_2 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (3)$$

and satisfy the commutation relations

$$[J_1, J_2] = -iJ_0 \quad [J_2, J_0] = iJ_1 \quad [J_0, J_1] = iJ_2. \quad (4)$$

These commutation relations define the abstract algebra  $su(1, 1)$ . In the several different realizations of this algebra considered below, we will always denote the generators by  $J_0, J_1, J_2$ . The Casimir operator, defined as  $C = J_1^2 + J_2^2 - J_0^2$ , commutes with the three generators. We introduce

$$J_{\pm} = J_1 \pm iJ_2. \quad (5)$$

In some instances, it will be more convenient to consider the group  $SL(2, \mathbb{R})$  consisting of matrices:

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \quad (6)$$

where the real numbers  $g_{11}, g_{12}, g_{21}, g_{22}$  verify the condition

$$g_{11}g_{22} - g_{12}g_{21} = 1. \quad (7)$$

The explicit form of the isomorphism between  $SU(1, 1)$  and  $SL(2, \mathbb{R})$  is given by

$$\begin{aligned} \gamma_1 &= \frac{1}{2}[g_{11} + g_{22} + i(g_{12} - g_{21})] \\ \gamma_2 &= \frac{1}{2}[g_{12} + g_{21} - i(g_{22} - g_{11})]. \end{aligned} \tag{8}$$

The affine group  $\mathcal{A}$  consists of elements  $(a, b)$ ,  $a > 0$  and  $b$  real, acting on an element  $x$  of the real line according to:  $x \rightarrow ax + b$ . It is isomorphic to the subgroup of elements of  $SL(2, \mathbb{R})$  given by

$$\begin{pmatrix} \sqrt{a} & b/\sqrt{a} \\ 0 & 1/\sqrt{a} \end{pmatrix} \quad a > 0 \quad b \in \mathbb{R} \tag{9}$$

and to the subgroup of  $SU(1, 1)$  consisting of matrices  $M(a, b)$  defined by

$$M(a, b) = \frac{1}{2\sqrt{a}} \begin{pmatrix} a + 1 + ib & i(a - 1 - ib) \\ -i(a - 1 + ib) & a + 1 - ib \end{pmatrix}. \tag{10}$$

The discrete series representations of  $SU(1, 1)$  (and  $SL(2, \mathbb{R})$ ), labelled by a number  $k \geq 1$  such that  $2k$  is an integer, are unitary and inequivalent. They will be described in three equivalent realizations.

**Representation  $T^k(\Gamma)$  in space  $\mathcal{H}_z$ :** In the space  $\mathcal{H}_z$  of functions  $f(z)$  that are analytic inside the unit circle, the operators  $T^k(\Gamma)$  representing the elements of the group  $SU(1, 1)$  are defined by

$$T^k(\Gamma)f(z) = (\gamma_2 z + \overline{\gamma_1})^{-2k} f\left(\frac{\gamma_1 z + \overline{\gamma_2}}{\gamma_2 z + \overline{\gamma_1}}\right). \tag{11}$$

This representation is unitary for the scalar product

$$(f, f') = \int_D \overline{f(z)} f'(z) (1 - |z|^2)^{2k-2} d\bar{z} dz \quad D = \{z, |z| < 1\}. \tag{12}$$

The generators of the algebra  $su(1, 1)$  are represented by the differential operators

$$J_0 = z\partial_z + k \tag{13}$$

$$J_+ = i(z^2\partial_z + 2kz) \tag{14}$$

$$J_- = -i\partial_z. \tag{15}$$

**Representation  $T^k$  in space  $H_w$ :** Another realization is more adapted to the group  $SL(2, \mathbb{R})$ . It is realized in the space  $H_w$  of holomorphic functions  $h(w)$  on the half-plane  $\text{Re}(w) > 0$ , equipped with the scalar product

$$(h, h') \equiv \int_{\text{Re}(w) > 0} \overline{h(\bar{w})} h'(w) (\text{Re}(w))^{2(k-1)} dw d\bar{w}. \tag{16}$$

The operator  $T^k$  representing an element  $g \in SL(2, \mathbb{R})$  is defined by

$$T^k(g)h(w) = (ig_{21}w + g_{11})^{-2k} h\left(\frac{g_{22}w - ig_{12}}{ig_{21}w + g_{11}}\right). \tag{17}$$

The space  $H_w$  of functions  $h(w)$  is isomorphic to the space  $\mathcal{H}_z$  of functions  $f(z)$  under the following transformation:

$$f(z) = 2(iz + 1)^{-2k} h\left(\frac{1 - iz}{1 + iz}\right). \tag{18}$$

The generators of  $su(1, 1)$  in representation  $T^k$  are found to be

$$\begin{aligned} J_0 &= \frac{1}{2}[(w^2 - 1)\partial_w + 2kw] \\ J_1 &= -\frac{1}{2}[(w^2 + 1)\partial_w + 2kw] \\ J_2 &= -i(w\partial_w + k). \end{aligned} \quad (19)$$

**Representation  $U^k$  in  $L_k^2(\mathbb{R}^+)$ :** There is another form of the representations of  $SL(2, \mathbb{R})$  and hence of  $SU(1, 1)$ , that has been studied in detail in [8] and that will be essential here. It acts in the Hilbert space  $L_k^2(\mathbb{R}^+)$  of functions  $\psi(y)$ ,  $y > 0$ , on the half-line with the scalar product

$$(\psi, \psi') = \int_0^\infty \overline{\psi(y)}\psi'(y)y^{1-2k} dy. \quad (20)$$

This space is applied isomorphically to  $H_w$  by a Laplace transformation written explicitly as

$$h(w) = \sqrt{\frac{(4\pi)^{2k-1}}{(2k-2)!}} \int_0^\infty \psi(y) e^{-2\pi wy} dy. \quad (21)$$

The representation  $U^k(g)$  of  $SL(2, \mathbb{R})$  in  $L_k^2(\mathbb{R}^+)$  that is equivalent to  $T^k$  can be written from there. In the following, we will only need the explicit form of the restriction of  $U^k$  to the affine subgroup of  $SU(1, 1)$ , which consists of elements  $M(a, b)$  defined in (10). It is equal to

$$U^k(M(a, b))\psi(y) = a^{(1-k)} e^{2i\pi by} \psi(ay). \quad (22)$$

This restriction is an irreducible representation of the affine group  $\mathcal{A}$ . Note that the representations of  $\mathcal{A}$  corresponding to different values of  $k$  are equivalent.

The generators of the representation  $U^k$  are obtained from (19) and (21):

$$J_0 = \frac{1}{4\pi} (-y\partial_y^2 + 2(k-1)\partial_y + 4\pi^2 y) \quad (23)$$

$$J_1 = -\frac{1}{4\pi} (-y\partial_y^2 + 2(k-1)\partial_y - 4\pi^2 y) \quad (24)$$

$$J_2 = i(y\partial_y + 1 - k). \quad (25)$$

### 3. Construction of the canonical basis for the algebra $su(1, 1)$

We now recall the construction of the canonical basis for the discrete series representation of the algebra  $su(1, 1)$  and give its explicit form in the spaces  $\mathcal{H}_z$ ,  $H_w$  and  $L_k^2(\mathbb{R}^+)$ . In those representations, the value of the Casimir operator is  $C = k(1-k)\hat{I}$  and the set of normalized vectors  $|k, m\rangle$  is defined by

$$\begin{aligned} J_0|k, m\rangle &= (k+m)|k, m\rangle \\ J_-|k, m\rangle &= \sqrt{[m]_k}|k, m-1\rangle \quad [m]_k \equiv m(2k+m-1) \\ J_+|k, m\rangle &= \sqrt{[m+1]_k}|k, m+1\rangle \end{aligned} \quad (26)$$

where  $m$  is a positive integer. The fundamental vector  $|k0\rangle$  is defined by

$$J_0|k0\rangle = k|k0\rangle \quad J_-|k0\rangle = 0. \quad (27)$$

The two equations are necessary as long as the representation space is not specified. The vectors  $|km\rangle$  are constructed in terms of  $|k0\rangle$  as

$$|km\rangle = \frac{1}{\sqrt{[m]_k!}} (J_+)^m |k0\rangle \quad [m]_k! \equiv \prod_{i=1}^m [i]_k = \frac{m!(2k+m-1)!}{(2k-1)!}. \quad (28)$$

**Canonical basis in  $\mathcal{H}_z$ :** Using the construction previously described, we obtain

$$\langle \bar{z} | km \rangle = \sqrt{\frac{(2k-1)}{\pi} \frac{\sqrt{[m]_k!}}{m!}} (iz)^m. \tag{29}$$

**Canonical basis in  $H_w$ :** The normalized states of the canonical basis in  $H_w$  are obtained from the inverse of transformation (18). They are equal to

$$\langle \bar{w} | km \rangle = \sqrt{\frac{(2k-1)}{\pi} \frac{\sqrt{[m]_k!}}{m!}} 2^{2k-1} \frac{(1-w)^n}{(w+1)^{2k+n}}. \tag{30}$$

**Canonical basis in  $L_k^2(\mathbb{R}^+)$ :** In the space  $L_k^2(\mathbb{R}^+)$ , the vectors of the canonical basis are the Laplace transforms of the previous ones. But the easiest way to obtain them is by a direct construction using the explicit expressions of the generators (23)–(25). The fundamental vector  $\langle y | k0 \rangle$  is defined again by conditions (27), which are written in space  $L_k^2(\mathbb{R}^+)$  as two compatible differential equations that reduce to

$$(y\partial_y + 2\pi y - 2k + 1)\langle y | k0 \rangle = 0. \tag{31}$$

The solution of (31) normalized for the scalar product (20) is

$$\langle y | k0 \rangle = \frac{(4\pi)^k}{\sqrt{(2k-1)!}} y^{2k-1} \exp(-2\pi y) \tag{32}$$

and verifies both equations of (27). Substituted in (28), this expression leads to

$$\langle y | km \rangle = \frac{(4\pi)^k}{\sqrt{(2k-1)! [m]_k!}} y^{2k-1} \exp(-2\pi y) \times P_m(y) \tag{33}$$

where  $P_m(y)$  are polynomials of degree  $m$  in  $y$ . These polynomials satisfy the following two equations that result from the action of the  $su(1, 1)$  generators, expressed in (23)–(25), on  $\langle y | km \rangle$ :

$$(y\partial_y + 2k + m - 4\pi y)P_m(y) = -P_{m+1}(y) \tag{34}$$

$$(-y\partial_y + m)P_m(y) = -[m]_k P_{m-1}(y). \tag{35}$$

These relations with the initial condition

$$P_0(y) = 1 \tag{36}$$

lead to the expression of the polynomials  $P_m(y)$

$$P_m(y) = (-1)^m m! L_m^{2k-1}(4\pi y) \tag{37}$$

where  $L_m^{2k-1}$  are the Laguerre polynomials [9].

#### 4. $SU(1, 1)$ coherent states in Perelomov’s parametrization

These coherent states are generated by action of the following elements of the group  $SU(1, 1)$ :

$$e^{\xi J_+ - \bar{\xi} J_-} \quad \left( \xi \equiv \frac{\tau}{2} e^{-i\varphi} \quad \tau \in \mathbb{R} \quad 0 \leq \varphi < 2\pi \right) \tag{38}$$

on the fundamental state. The result is

$$|\zeta\rangle \equiv e^{\xi J_+ - \bar{\xi} J_-} |k0\rangle = (1 - |\zeta|^2)^k \sum_{m \geq 0} \sqrt{[m]_k!} \frac{\zeta^m}{m!} |km\rangle \tag{39}$$

where

$$\zeta = \tanh \frac{\tau}{2} \exp(-i\varphi). \tag{40}$$

Since  $|k0\rangle$  is an eigenstate of  $J_0$ , the set of coherent states will depend only on the quotient of  $SU(1, 1)$  by the rotation group  $\mathcal{R}$ . Such a quotient is isomorphic to the upper sheet of the hyperboloid  $n_0^2 - n_1^2 - n_2^2 = 1$ , parametrized by  $(\tau, \varphi)$  in the following way:

$$\vec{n} = (\cosh \tau, \sinh \tau \cos \varphi, \sinh \tau \sin \varphi) \quad (41)$$

and to its stereographic projection onto the inside of the unit disc parametrized by  $\zeta$  given in (40).

The coherent states thus obtained verify the completeness relation

$$\frac{2k-1}{\pi} \int_D \frac{d^2\zeta}{(1-|\zeta|^2)^2} |\zeta\rangle\langle\zeta| = 1 \quad D = \{\zeta, |\zeta| < 1\} \quad (42)$$

and form an overcomplete set. The whole set of rays defined by the coherent states  $|\zeta\rangle$  is stable under the action of  $SU(1, 1)$ . In particular, the rotation subgroup acts on such a state through the operator  $\exp(-i\theta J_0)$  as

$$\exp(-i\theta J_0)|\zeta\rangle = e^{-ik\theta} |\zeta e^{-i\theta}\rangle. \quad (43)$$

The explicit form of the Perelomov coherent states in the different spaces considered above results from expressions (29), (30) and (33) of the canonical basis.

**Coherent states in  $\mathcal{H}_z$ :**

$$\langle \bar{z} | \zeta \rangle = \sqrt{\frac{2k-1}{\pi}} \frac{(1-|\zeta|^2)^k}{(1-i\zeta z)^{2k}}. \quad (44)$$

**Coherent states in  $H_w$ :**

$$\langle \bar{w} | \zeta \rangle = \sqrt{\frac{(2k-1)}{\pi}} 2^{2k-1} \frac{(1-|\zeta|^2)^k}{(w+1-\zeta(1-w))^{2k}}. \quad (45)$$

**Coherent states in  $L_k^2(\mathbb{R}^+)$ :** The computation of the coherent states in the space  $L_k^2(\mathbb{R}^+)$  uses expression (33) of the canonical basis in that space so that (39) leads to

$$\langle y | \zeta \rangle = \sqrt{\frac{(4\pi)^{2k}}{(2k-1)!}} (1-|\zeta|^2)^k y^{2k-1} \exp(-2\pi y) \sum_{m \geq 0} \frac{\zeta^m}{m!} P_m(y). \quad (46)$$

This expression involves the generating function of the polynomials  $P_m$  which is computed from that of the Laguerre polynomials:

$$\mathcal{P}(\zeta, y) \equiv \sum_{m \geq 0} \frac{\zeta^m}{m!} P_m(y) = (1+\zeta)^{-2k} \exp\left(4\pi y \frac{\zeta}{\zeta+1}\right). \quad (47)$$

The Perelomov coherent states expressed in the space  $L_k^2(\mathbb{R}^+)$  are thus equal to

$$\langle y | \zeta \rangle = \sqrt{\frac{(4\pi)^{2k}}{(2k-1)!}} \frac{(1-|\zeta|^2)^k}{(1+\zeta)^{2k}} y^{2k-1} \exp\left(2\pi y \frac{\zeta-1}{\zeta+1}\right) \quad |\zeta| < 1. \quad (48)$$

When  $k = 1$ , these functions coincide (up to normalization) with coherent states introduced in [10] for the Morse problem.

## 5. Parametrization of the $SU(1, 1)$ coherent states in terms of the affine group

The quotient space of  $SU(1, 1)$  by the rotation group is a group isomorphic to the affine group  $\mathcal{A}$ . This is most easily seen when working with  $SL(2, \mathbb{R})$  since any matrix  $g$  defined in (6) can be uniquely decomposed into the product of a matrix of the affine subgroup by a

rotation matrix as

$$g = \begin{pmatrix} h_{11} & h_{12} \\ 0 & h_{11}^{-1} \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}. \tag{49}$$

This property will now be exploited systematically.

5.1. Affine group interpretation of Perelomov states

At the algebra level, the affine group generators are given in terms of those of  $SU(1, 1)$  by the relations

$$A \equiv J_0 + J_1 \quad B \equiv J_2 \tag{50}$$

leading to the commutation relation

$$[B, A] = iA. \tag{51}$$

The action on the space  $L_k^2(\mathbb{R}^+)$  is

$$A = 2\pi y \tag{52}$$

$$B = i(y\partial_y + 1 - k). \tag{53}$$

The construction of the Perelomov coherent states will now be performed in terms of generators  $J_0$  and  $A, B$ .

The use of relations (50) allows us to replace equations (27) defining the fundamental state  $\langle y|k0\rangle$  by the equivalent set

$$J_0 \langle y|k0\rangle = k \langle y|k0\rangle \quad (A - iB) \langle y|k0\rangle = k \langle y|k0\rangle. \tag{54}$$

Here the problem is set up in the Hilbert space  $L_k^2(\mathbb{R}^+)$  with a specific value of  $k$  and the second equation, involving the affine group generators, is sufficient to determine the function  $\langle y|k0\rangle$ . Next, we introduce the matrix  $D(\tau, \varphi)$  of  $SU(1, 1)$  corresponding to the element  $e^{\xi J_+ - \bar{\xi} J_-}$  defined in (38):

$$D(\tau, \varphi) = \begin{pmatrix} \cosh(\tau/2) & -ie^{-i\varphi} \sinh(\tau/2) \\ ie^{i\varphi} \sinh(\tau/2) & \cosh(\tau/2) \end{pmatrix}. \tag{55}$$

The Perelomov coherent states are defined as displaced from  $\langle y|k0\rangle$  by operator  $U^k(D(\tau, \varphi))$ . But since  $\langle y|k0\rangle$  is an eigenstate of the rotation operator, it is possible to perform a rotation on  $\langle y|k0\rangle$  before applying  $U^k(D(\tau, \varphi))$  and still obtain a state belonging to the same ray. We will take advantage of this fact to define the coherent states by an affine transformation.

Multiplying  $D(\tau, \varphi)$  on the right by the rotation matrix  $\Gamma_\theta$  defined by the operator  $e^{-i\theta J_0}$ :

$$\Gamma_\theta = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \tag{56}$$

we can determine  $\theta$  (as a function of  $\tau$  and  $\varphi$ ) so as to obtain an element of the affine group  $\mathcal{A}$ :

$$D(\tau, \varphi)\Gamma_\theta \equiv M(a, b) \tag{57}$$

where  $M(a, b)$  is defined in (10). This relation gives a one-to-one correspondence between parameters  $(\tau, \varphi)$  and  $(a, b)$ . Using definition (40) of  $\zeta$  in terms of  $(\tau, \varphi)$  leads to the expressions

$$\zeta = \frac{1 - a + ib}{1 + a - ib} \quad e^{-i\theta} = \frac{1 + a + ib}{1 + a - ib}. \tag{58}$$



Coherent states  $\langle y|ab\rangle$  are now defined as transforms of  $\langle y|k0\rangle$  by the operator  $U_k(M(a, b))$ . Their explicit form is obtained using (22) and (32):

$$\langle y|ab\rangle \equiv U^k(M(a, b))\langle y|k0\rangle = \frac{(4\pi)^k}{\sqrt{(2k-1)!}} a^k \exp(2\pi(-a+ib)y) y^{2k-1}. \quad (59)$$

They are related to Perelomov states by

$$|ab\rangle = \left( \frac{1+\zeta}{1+\bar{\zeta}} \right)^k |\zeta\rangle \quad (60)$$

where the expressions of the parameters  $(a, b)$  in terms of  $\zeta$  are deduced from (58).

The completeness relation in variables  $(a, b)$  is obtained from (42) and reads

$$\frac{2k-1}{4\pi} \int_0^\infty \frac{da}{a^2} \int_{-\infty}^\infty db |ab\rangle \langle ab| = 1. \quad (61)$$

As a result, the Perelomov coherent states can be characterized either by  $(a, b)$  or  $\zeta$ . They form an overcomplete basis of the space  $L_k^2(\mathbb{R}^+)$  that is constructed from the fundamental state  $\langle y|k0\rangle$  by applying an affine group transformation.

### 5.2. Conditions for the affine group wavelets to be $SU(1, 1)$ coherent states

General coherent states associated with the affine group, also known as wavelets, may be constructed using fundamental states different from  $|k0\rangle$  [1, 2]. In fact, choosing an element  $\sigma_0(y) \in L_k^2(\mathbb{R}^+)$  ('mother wavelet'), we can construct the family  $\sigma_{a,b}(y)$  by application of representation (22) of group  $\mathcal{A}$  as

$$\sigma_{ab}(y) \equiv a^{1-k} e^{2i\pi by} \sigma_0(ay). \quad (62)$$

Any state  $\psi(y)$  belonging to  $L_k^2(\mathbb{R}^+)$  can be developed on the family  $\{\sigma_{ab}(y)\}$  with coefficients ('wavelet coefficients') given by

$$C(a, b) = (\psi(y), \sigma_{ab}(y)) \quad (63)$$

where  $(,)$  denotes the scalar product (20). A direct computation shows that the state  $\psi(y)$  can be reconstructed from its coefficients  $C(a, b)$  provided the mother wavelet  $\sigma_0(y)$  satisfies the condition:

$$\int_0^\infty |\sigma_0(y)|^2 y^{-2k} dy < \infty. \quad (64)$$

This so-called admissibility condition is usually written with  $k = 1/2$ . Recall that it is possible to choose a particular value of  $k$  when considering the affine group alone because of the equivalence of its representations for different values of  $k$ .

Thus there exists an infinite family of overcomplete bases constructed with the affine group for space  $L_k^2(\mathbb{R}^+)$ . However, if the invariance (up to a phase) by  $SU(1, 1)$  is required, the basic state  $\sigma_0(y)$  must be an eigenstate of the rotation operator  $J_0$ . This restricts the choice to  $\sigma_0(y) = \langle y|km\rangle$ .

### 5.3. Morse coherent states

Group theoretical arguments have led to the use of  $SU(1, 1)$  coherent states in the problem of the Morse oscillator [7]. But several different representations are then required for a complete description. A more satisfactory family of coherent states has been introduced as eigenstates of an annihilation operator by Benedict and Molnar in [10] and shown to be related to affine coherent states [11]. This study allows us to find the exact relation between the two sets.

The Morse potential considered in [10] has the form

$$V(x) = \left(s + \frac{1}{2} - e^{-x}\right)^2 \quad (65)$$

where  $s$  is a real parameter such that  $s > 1/2$ . The relevant Hilbert space of the problem is  $L_k^2(\mathbb{R}^+)$  for  $k = 1$  and the system of coherent states can be constructed by displacement of the fundamental state

$$\phi_0(y) = \frac{(4\pi)^s}{\sqrt{\Gamma(2s)}} y^s e^{-2\pi y}. \quad (66)$$

When  $s = 1$ , this state coincides with the state  $\langle y|k0\rangle$  defined in (32) and the corresponding coherent states are identical with Perelomov states  $|\zeta\rangle$ . When  $s \neq 1$ , the state (66) is no longer invariant by the subgroup of rotations but, as recalled in section 5.2, it can still be used to construct affine coherent states. The latter are, up to a phase, equal to the states considered in [10].

In conclusion, Perelomov coherent states for the discrete series representation of  $SU(1, 1)$  are a subset of the coherent states considered in [10] for a Morse potential problem.

## 6. Consequences of the new characterization of the $SU(1, 1)$ coherent states

The properties of coherent states obtained by applying the displacement operator on a fiducial state come directly from those of the latter. In particular, when the fiducial state is  $|k0\rangle$ , the corresponding coherent states have minimal properties and satisfy equations derived from (27). The explicit results are most easily derived using the parametrization in terms of the affine group, as shown below.

Let  $O_1$  and  $O_2$  be two self-adjoint operators and let  $\langle O_i \rangle$ ,  $i = 1, 2$  denote their mean values in an arbitrary state  $|\psi\rangle$ . Introduce the centred operator  $\overline{O}_i$  as

$$\overline{O}_i = O_i - \langle O_i \rangle \quad (67)$$

and define the mean square deviations

$$\Delta_i = \langle \overline{O}_i^2 \rangle \quad (68)$$

and the correlation

$$\Delta_{12} = \langle \overline{O}_1 \overline{O}_2 + \overline{O}_2 \overline{O}_1 \rangle. \quad (69)$$

Writing that the norm of the state  $(\overline{O}_1 + i\lambda \overline{O}_2)|\psi\rangle$  is positive for every complex value of  $\lambda$  leads to the generalized uncertainty relations:

$$4\Delta_1\Delta_2 - \Delta_{12}^2 \geq (\langle i[O_1, O_2] \rangle)^2. \quad (70)$$

Starting from a real  $\lambda$ , one obtains the more usual relation:

$$4\Delta_1\Delta_2 \geq (\langle i[O_1, O_2] \rangle)^2. \quad (71)$$

The equality in (70) is obtained for states  $|\psi\rangle$  verifying

$$(\overline{O}_1 + i\lambda \overline{O}_2)|\psi\rangle = 0 \quad (72)$$

for complex values of  $\lambda$ . When  $\lambda$  is real, the correlation  $\Delta_{12}$  vanishes and the corresponding state minimizes the stricter relation (71).

This general scheme is now applied to the generators  $A, B$  of the affine group and to the coherent states  $|ab\rangle$ .

Property (54) implies that the affine coherent states verify the following relation:

$$M(a, b)(A - iB)M(a, b)^{-1}|a, b\rangle \equiv ((a - ib)A - iB)|a, b\rangle = k|a, b\rangle \quad (73)$$

which allows us to compute the mean values of the affine generators:

$$\langle A \rangle = ka^{-1} \quad \langle B \rangle = -kba^{-1}. \quad (74)$$

Relations (73) and (74) lead to the equations characterizing the coherent states  $|ab\rangle$ :

$$(\bar{B} + i(a - ib)\bar{A})|ab\rangle = 0. \quad (75)$$

This equation is of the form (72). In the present case, the parameter  $\lambda$  has a definite value  $\lambda = a - ib$  depending on the state  $|ab\rangle$ . The correlation  $\Delta_{12}$  between  $A$  and  $B$  vanishes only for  $b = 0$ . However, for each state, it is possible to introduce uncorrelated operators  $A$  and  $a^{-1}(B + bA)$ .

The exploitation of these results will be performed in terms of  $\zeta$ . Equation (73) becomes

$$((1 - \zeta)J_0 - \zeta J_+ + J_- - k(\zeta + 1))|\zeta\rangle = 0. \quad (76)$$

Let us denote  $\tilde{J}_i \equiv \exp(\xi J_+ - \bar{\xi} J_-) J_i \exp(-\xi J_+ + \bar{\xi} J_-)$ . The coherent states verify two relations resulting from the properties of the fundamental state (27):

$$\tilde{J}_0|\zeta\rangle \equiv \vec{n} \cdot \vec{J}|\zeta\rangle = k|\zeta\rangle \quad (77)$$

and

$$\tilde{J}_-|\zeta\rangle \equiv (J_- - 2\zeta J_0 + \zeta^2 J_+)|\zeta\rangle = 0. \quad (78)$$

The combination of (78) and (76) gives two simpler equations

$$(J_0 - \zeta J_+ - k)|\zeta\rangle = 0 \quad (79)$$

and

$$(\zeta J_0 - J_- + k\zeta)|\zeta\rangle = 0. \quad (80)$$

To compute the mean values of  $\langle \vec{J} \rangle$ , we multiply these equations on the left by  $\langle \zeta |$  and find

$$\begin{aligned} \langle J_0 \rangle &= k \frac{1 + |\zeta|^2}{1 - |\zeta|^2} = k \cosh \tau \\ \langle J_1 \rangle &= k \frac{\zeta + \bar{\zeta}}{1 - |\zeta|^2} = k \sinh \tau \cos \varphi \\ \langle J_2 \rangle &= ik \frac{\zeta - \bar{\zeta}}{1 - |\zeta|^2} = k \sinh \tau \sin \varphi. \end{aligned} \quad (81)$$

Thus the two vectors  $\langle \vec{J} \rangle$  and  $\vec{n}$ , defined in (41), have the same direction:

$$\langle \vec{J} \rangle = k\vec{n}. \quad (82)$$

Due to equations (79), (80) and results (81), the coherent states  $|\zeta\rangle$  verify the following equations:

$$\left( \bar{J}_1 + i \frac{\zeta^2 + 1}{\zeta^2 - 1} \bar{J}_2 \right) |\zeta\rangle = 0 \quad (83)$$

$$\left( \bar{J}_0 - \frac{2\zeta}{\zeta^2 + 1} \bar{J}_1 \right) |\zeta\rangle = 0 \quad (84)$$

$$\left( \bar{J}_0 + i \frac{2\zeta}{\zeta^2 - 1} \bar{J}_2 \right) |\zeta\rangle = 0. \quad (85)$$

The interpretation of these equations for real  $\zeta$  shows that the coherent states  $|\zeta\rangle$  minimize the usual uncertainty relation (71) for the pairs  $(J_1, J_2)$  and  $(J_0, J_2)$ . These states are associated

with the section of the upper sheet of the hyperboloid by the plane  $(n_0, n_1)$ . Similarly, the section by the plane  $(n_0, n_2)$  corresponds to coherent states  $|\zeta\rangle$  with purely imaginary  $\zeta$  that minimize relation (71) for the pairs  $(J_1, J_2)$  and  $(J_0, J_1)$ .

For other values of  $\zeta$ , the operators  $(J_i, J_j)$  are correlated. However, it is always possible to construct uncorrelated operators in the form  $J_i + \lambda J_j$  and  $J_i + \mu J_j$ , where  $\lambda$  and  $\mu$  depend on the parameter  $\zeta$ .

## 7. Conclusion

Realizing the discrete series representation of  $SU(1, 1)$  labelled by  $k$  and the corresponding representation of the affine group in the same Hilbert space, we have been able to make a precise comparison of the coherent states attached to the two groups. The Perelomov coherent states constructed either on  $|k0\rangle$  or  $|km\rangle$  have been found identical, up to a phase, to the special families of affine coherent states or wavelets. Conversely, affine group coherent states obtained from a basic state that is invariant under rotations form an overcomplete basis that is invariant as a whole under the  $SU(1, 1)$  representation.

The characterization of the rays in terms of affine wavelets has several advantages: minimal properties of the states and characteristic equations are easily obtained. More fundamental is the result that the set of  $SU(1, 1)$  coherent states is strictly invariant by the action of the affine group representation, while it is invariant only up to a phase by the action of  $SU(1, 1)$ . These results stress the importance of determining the invariance group of a problem to take full advantage of the properties of the system of coherent states.

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